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DYNAMICS OF LINEAR ANISOTROPIC RODS

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Abstract—We obtain a one-dimensional model for the dynamics of a rod-like body as an exact consequence of three-dimensional linear anisotropic elasticity, by means of the internal constraint of indeformability of the cross-section in its plane. The model takes into account *flexure*, *extension*, *torsion* and *warping* deformations, which are coupled due to the anisotropy; if we assume that the symmetry group of the material comprising the rod contains the reflections upon the cross-section, the one-dimensional equations of motion split into three independent groups concerned, respectively, with *extension*, *flexure* and *torsion-warping* deformations. This result generalizes the equations obtained by Green *et al.* [(1967) *Arch. Rational Mech. Analys.* **25**, 285–298] and those originally proposed by Timoshenko [(1921) *Philos. Mag.* **43**, 125–131] and Vlasov [(1961) *Thin-walled Elastic Beams*. Israel Program for Scientific Translation, Tel Aviv] in the isotropic case.

1. INTRODUCTION

In recent years, the structural mechanics of anisotropic bodies has become of growing importance for its applications to the analysis of composite and smart structures, as well as to bone mechanics, plant mechanics, etc. It is noteworthy that for the three-dimensional problem there is a large literature, from the work of Voigt (1910) to the books of Lekhnitskii (1981) and Gurtin (1972), whereas when we deal with one-dimensional problems the material is assumed to be either isotropic (e.g. Timoshenko, 1921; Vlasov, 1961; Reissner 1983a) or with anisotropy in the plane of the cross-section as in Green *et al.* (1967) or Reissner (1983b). The generalization we get with such anisotropy is limited because the *extension*, *flexure* (i.e. bending by terminal transverse loads), and *torsion-warping* deformations of the rod are uncoupled as in the isotropic case.

In this paper we wish to obtain, from the equations of three-dimensional linear anisotropic elasticity, a one-dimensional model for the dynamics of anisotropic rods which takes in account the couplings between extension, flexure and torsion-warping deformations due to the anisotropy. The reduction from the three-dimensional model is obtained, as in Daví (1992) by assuming that the body occupies a *rod-like region*, and that in each point during the motion, the internal constraint of indeformability of the cross-section in its plane prevails; this is a non-standard manner to formulate the concept of *thinness* of the rod, which rules out arguments like "smallness" of kinematical terms and similar. As customary in linear elasticity, the presence of an internal constraint splits the stress into a reactive part, whose role is to maintain the constraint and does no work, and into an active part; moreover, the constitutive relation for the active part must comply with the prescribed constraint as indicated by Podio-Guidugli and Vianello (1992).

Unlike Daví (1992) we are concerned with straight rods, because we are interested in the kinematical couplings induced by the anisotropy, rather than by the curvature and torsion of the axis of the rod; however the method we use can readily be applied to curved rods.

We assume the internal constraint as an exact mathematical restriction on the admissible motion, and then we are able to obtain, by using the constraint equation and the kinematical compatibility equations, the following representation for the admissible displacement field:

$$\mathbf{u}(\mathbf{x},\zeta;t) = \mathbf{v}(\zeta;t)+\mathbf{\Lambda}(\zeta;t)\mathbf{x}+\omega(\zeta;t)\Phi(\mathbf{x})\mathbf{e},$$

where $\mathbf{x}$ and ζ are respectively the cross-sectional and axial coordinates, t is time, $\mathbf{e}$ is the direction and $\mathbf{v}$ the displacement of the axis, $\mathbf{\Lambda}$ is a skew-symmetric tensor which represents the rigid rotation of the cross-section and Φ is the warping, with amplitude ω, of the cross-section. Such a representation was first proposed by Vlasov (1961), as *a priori* given suitable approximation for the displacement field: here we show that it is the only *exact* solution for the system of differential equations which is obtained by taking into account both the constraint and kinematical compatibility equations.

In the current literature, the warping function Φ is assumed as the solution of the Saint-Venant's pure torsion problem; since the three-dimensional problem we are dealing with is different from the Saint-Venant's one, in this paper we propose a warping function which is consistent with our three-dimensional formulation.

Once we obtain the admissible displacement field, we write the admissible strain and the active stress; by introducing stress and couple resultants over the cross-section and the bi-shear and bi-moment characteristics we get, from the Hamilton–Kirchhoff principle, the one-dimensional balance equations for the dynamics of anisotropic rods.

If we assume that the symmetry group of the material comprising the rod contains the reflections upon the cross-section we are back to the case of anisotropy limited to the cross-section and we are dealing with *monocline* materials and rods. As pointed out by Daví and Tiero (1994) the qualitative behaviour is the same as in the isotropic case and the equations of motion split into three independent groups concerned respectively with extension, flexure and torsion-warping deformations; this result generalizes the equations obtained by Green *et al.* (1967) (within the context of a director theory) for *orthotropic-rhombic* rods and those originally proposed by Timoshenko (1921) and Vlasov (1961) for *isotropic* rods.

2. KINEMATICS OF LINEAR THIN RODS

2.1. *Three-dimensional rod-like bodies*

We present a *three-dimensional rod-like body* $\mathscr{R}$, modelled on a *straight line* $\mathscr{L}$, a three-dimensional body which occupies the cylindrical region $\mathscr{R}\equiv\mathscr{S}\times\mathscr{I}$ of the euclidean point space $\mathscr{E}$ (with an associated vector space $\mathscr{V}$) such that the typical point $p\in\mathscr{R}$ be

$$p = x+\zeta\mathbf{e}, \quad (x,\zeta)\in\mathscr{S}\times\mathscr{I}, \tag{1}$$

where the *cross-section* $\mathscr{S}$ is an open, regular planar two-dimensional domain with boundary $\partial\mathscr{S}$ lying in a plane orthogonal to $\mathscr{L}$, $\mathscr{I}\equiv]0,L[$ is an open interval of the real axis $\mathbb{R}$, and $\mathbf{e}\in\mathscr{V}$ the tangent unit vector to $\mathscr{L}$. Let o be a given point of $\mathscr{S}$ and $\mathbf{x}(x) = x-o$ be the position vector of x, then:

$$\mathbf{p}(\mathbf{x},\zeta) = p-o = \mathbf{x}+\zeta\mathbf{e}, \quad \mathbf{x}\cdot\mathbf{e} = 0, \quad \zeta\in]0,L[. \tag{2}$$

The boundary $\partial\mathscr{R}$ consists of three complementary regular subsets, the *mantle* $\mathscr{M}\equiv\partial\mathscr{S}\times\mathscr{I}$ and the two *bases*, i.e. the terminal cross-sections $\mathscr{S}\times\{0\}$ and $\mathscr{S}\times\{L\}$; we call $\mathscr{L}$ the *axis* of the rod.

Let $\mathbf{u}$ be a *motion* for $\mathscr{R}$, say a vector field $\mathbf{u}(\mathbf{p};t) = \mathbf{u}(\mathbf{x},\zeta;t)$ on $\mathscr{R}\times[0,\tau)$ where $[0,\tau)$ is a given time interval; we find it useful to introduce the following decomposition:

$$\mathbf{u}(\mathbf{x},\zeta;t) = \hat{\mathbf{u}}(\mathbf{x},\zeta;t)+u(\mathbf{x},\zeta;t)\mathbf{e}, \quad \hat{\mathbf{u}}\in\mathscr{W}, \quad \mathbf{u}\in\mathbb{R}, \tag{3}$$

with $\mathscr{W}\equiv\{\mathbf{a}\in\mathscr{V}|\mathbf{a}\cdot\mathbf{e} = 0\}$. Let $\mathbf{E}(\mathbf{p};t) = \mathbf{E}(\mathbf{x},\zeta;t)$ be the infinitesimal strain field defined as

$$\mathbf{E} = \operatorname{sym} \nabla \mathbf{u} = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{T}), \tag{4}$$

defined by $\mathscr{R} \times [0,\tau)$ and taking values in sym, the space of the second-order symmetric tensors; we introduce the decomposition:

$$\mathbf{E}(\mathbf{x}, \zeta ; t) = \hat{\mathbf{E}}(\mathbf{x}, \zeta ; t) + \operatorname{sym} \gamma(\mathbf{x}, \zeta ; t) \otimes \mathbf{e} + \varepsilon(\mathbf{x}, \zeta ; t)\, \mathbf{e} \otimes \mathbf{e}, \tag{5}$$

with the *shear formation* field $\gamma \in \mathscr{W}$, the *axial extension* field $\varepsilon \in \mathbb{R}$, and the *plane strain* field $\hat{\mathbf{E}} \in \mathscr{K}$, where

$$\mathscr{K} \equiv \{\mathbf{A} \in \operatorname{sym} \mid \mathbf{A}\mathbf{e} = \mathbf{0}\}. \tag{6}$$

The decompositions (3) and (5) reflect on (4) which becomes:

$$\begin{aligned} \hat{\mathbf{E}} &= \operatorname{sym} \nabla_{x} \hat{\mathbf{u}}, \\ \gamma &= \nabla_{x} u + \hat{\mathbf{u}}_{,\zeta}, \\ \varepsilon &= u_{,\zeta}; \end{aligned} \tag{7}$$

likewise, (5) splits the kinematical compatibility equation curl curl $\mathbf{E} = \mathbf{0}$ into

$$\begin{aligned} \nabla_{x} \nabla_{x} \varepsilon + \hat{\mathbf{E}}_{,\zeta\zeta} &= (\nabla_{x} \gamma + \nabla_{x} \gamma^{T})_{,\zeta}, \\ \nabla_{x}(\operatorname{curl} \gamma) &= (\operatorname{curl} \hat{\mathbf{E}})_{,\zeta}, \\ \operatorname{curl}_{x} \operatorname{curl}_{x} \hat{\mathbf{E}} &= 0, \end{aligned} \tag{8}$$

where ∇_{x} denotes the gradient operator with respect to $\mathbf{x}$ and curl_{x} is defined, for all constant vectors $\mathbf{a} \in \mathscr{W}$, as:

$$\operatorname{curl}_{x} \hat{\mathbf{E}} \cdot \mathbf{a} = \operatorname{curl}(\hat{\mathbf{E}}\mathbf{a}) \cdot \mathbf{e}. \tag{9}$$

Let $\mathbf{S}(\mathbf{p}; t) = \mathbf{S}(\mathbf{x}, \zeta ; t) \in \operatorname{sym}$ be the Cauchy stress tensor field, defined on $\mathscr{R} \times [0,\tau)$: in the same way as we did for the strain field, we introduce the decomposition

$$\mathbf{S}(\mathbf{x}, \zeta ; t) = \hat{\mathbf{S}}(\mathbf{x}, \zeta ; t) + \operatorname{sym} \tau(\mathbf{x}, \zeta ; t) \otimes \mathbf{e} + \sigma(\mathbf{x}, \zeta ; t) \mathbf{e} \otimes \mathbf{e}, \tag{10}$$

where the *plane stress* field is $\hat{\mathbf{S}} \in \mathscr{K}$, the *shear tension* field $\tau \in \mathscr{W}$, and the *axial tension* field $\sigma \in \mathbb{R}$.

We assume that $\mathscr{R}$ consists of a linearly elastic material, i.e. that the stress $\mathbf{S}$ is obtained from the strain $\mathbf{E}$ by means of a symmetric, positive definite, linear mapping $\mathbb{C}(\mathbf{p}) = \mathbb{C}(\mathbf{x}, \zeta)$, the *elasticity tensor*:

$$\mathbf{S} = \mathbb{C}[\mathbf{E}]. \tag{11}$$

We remember here that a material is *anisotropic* if its *symmetry group* $\mathscr{G}_{\mathbf{p}}$ at a given point $\mathbf{p}$, defined as the set of all orthogonal tensors $\mathbf{Q}$ which obey:

$$\mathbf{Q}\mathbb{C}[\mathbf{E}]\mathbf{Q}^{T} = \mathbb{C}[\mathbf{Q}\mathbf{E}\mathbf{Q}^{T}], \quad \forall \mathbf{E} \in \operatorname{sym}, \tag{12}$$

is a proper subgroup of the orthogonal group, whereas it is *isotropic* when $\mathscr{G}_{\mathbf{p}}$ is the orthogonal group.

The decompositions (5) and (10) induce the following one on the stress–strain relation (11):

$$\hat{\mathbf{S}} = \hat{\mathbb{C}}[\hat{\mathbf{E}}] + \mathbb{L}\gamma + \mathbf{D}\varepsilon,$$
$$\tau = \mathbb{L}^T\hat{\mathbf{E}} + \mathbf{G}\gamma + \mathbf{h}\varepsilon,$$
$$\sigma = \mathbf{D}\cdot\hat{\mathbf{E}} + \mathbf{h}\cdot\gamma + E\varepsilon; \qquad (13)$$

here $\hat{\mathbb{C}}$ maps $\mathscr{K}$ into itself, $\mathbb{L}$ maps $\mathscr{W}$ into $\mathscr{K}$ and $\mathbf{D}, \mathbf{G} \in \mathscr{K}, \mathbf{h} \in \mathscr{W}, E \in \mathbb{R}$; the positive definiteness of $\mathbb{C}$ implies in turn the positive definiteness of $\hat{\mathbb{C}}$, $\mathbf{G}$ and $E > 0$.

Up to this point we have simply reflected the structure of Cartesian product underlying the rod-like region into the motion, strain and stress fields, as well as into the kinematical field equations and constitutive relation for the body occupying $\mathscr{R}$; now we declare that $\mathscr{R}$ is a *thin* rod.

In this regard, it is customary to define the *thinness* in terms of the geometry of the rod (typically if d denotes the diameter of the cross-section $\mathscr{S}$ we say a rod is thin when dL^{-1} is "small" in some sense) and then to reflect this into the kinematics by assuming that some deformations are negligible when the rod is thin [e.g. Love (1927), §. 257 and Hay (1942)]; here we want to reverse the role of geometry and kinematics by assuming that

> a rod-like body $\mathscr{R}$ is *thin* if, in every motion, the plane deformations of its cross-section $\mathscr{S}$ are negligible.

i.e. we give a phenomenological description of the thinness rather than a geometrical one.

In the notation we introduced, this statement is equivalent to the following restriction on the strain field:

$$\hat{\mathbf{E}}(\mathbf{x}, \zeta; t) = 0, \quad \forall(\mathbf{x}, \zeta; t) \in \mathscr{S} \times \mathscr{I} \times [0, \tau). \qquad (14)$$

2.2. *Kinematics*

The kinematical assumption (14) is an internal constraint which restricts the possible deformations of the three-dimensional rod-like body $\mathscr{R}$. As is customary in linear elasticity [cf. Gurtin and Podio-Guidugli (1973)], an internal constraint is the prescription of a linear manifold $\mathscr{Y}$ of admissible strain; moreover, the presence of a frictionless internal constraint splits addictively the stress $\mathbf{S}$ into a reactive part $\mathbf{S}^{(R)}$, whose role is to maintain the constraint and does not work in any admissible motion, and an active part $\mathbf{S}^{(A)}$:

$$\mathbf{S} = \mathbf{S}^{(A)} + \mathbf{S}^{(R)},$$
$$\mathbf{S}^{(A)} = \mathbb{C}[\mathbf{E}], \quad \forall \mathbf{E} \in \mathscr{Y},$$
$$\mathbf{S}^{(R)} \in \mathscr{Y}^{\perp}, \qquad (15)$$

where $\mathscr{Y}^{\perp}$ denotes the orthogonal complement of $\mathscr{Y}$ in sym; as in Podio-Guidugli and Vianello (1992), we require the elasticity tensor $\mathbb{C}$ to reflect the maximal material symmetry compatible with the prescribed constraint, which implies:

$$\mathbb{C}[\mathbf{V}] = \mathbf{0}, \quad \forall \mathbf{V} \in \mathscr{Y}^{\perp}. \qquad (16)$$

When (14) plays the role of an internal constraint, then the linear manifold of admissible strain for the rod-like body $\mathscr{R}$ is

$$\mathscr{Y} \equiv \{\mathbf{E} \in \text{sym} \,|\, \mathbf{E}\cdot\hat{\mathbf{V}} = 0, \forall \hat{\mathbf{V}} \in \mathscr{K}\}, \qquad (17)$$

with $\hat{\mathbf{V}}$ independent on both $\mathbf{p}$ and t; accordingly, $\mathscr{Y}^{\perp} \equiv \mathscr{K}$ and the reactive stress is given by:

$$\mathbf{S}^{(R)} = \hat{\mathbf{S}}, \tag{18}$$

whereas, in order to comply with condition (16), the elasticity tensor (13) must obey:

$$\hat{\mathbb{C}} = \mathbf{0}, \quad \mathbb{L} = \mathbf{0}, \quad \mathbf{D} = \mathbf{0}. \tag{19}$$

To obtain a representation for the motion fields associated with the admissible strain we notice that, by using (14) in $(7)_1$, the transverse component $\hat{\mathbf{u}}$ can be arrived at by direct integration:

$$\hat{\mathbf{u}}(\mathbf{x}, \zeta ; t) = \hat{\mathbf{v}}(\zeta ; t) + \theta(\zeta ; t)\mathbf{e} \times \mathbf{x}, \tag{20}$$

where $\theta \in \mathbb{R}$ and $\hat{\mathbf{v}} \in \mathscr{W}$. We make use of (14) in the kinematical compatibility eqns (8), which reduce to

$$\nabla_x \nabla_x \varepsilon = (\nabla_x \gamma + \nabla_x \gamma^{\mathrm{T}})_{,\zeta}, \quad \nabla_x (\operatorname{curl} \gamma) = \mathbf{0}; \tag{21}$$

and that can be now easily integrated to obtain the axial component u of the motion. First of all, from eqn $(21)_2$ we get

$$\gamma(\mathbf{x}, \zeta ; t) = \gamma_0(\zeta ; t) + \alpha(\zeta ; t)\mathbf{e} \times \mathbf{x} + \omega(\zeta ; t)\nabla_x \Phi(\mathbf{x}), \tag{22}$$

with Φ an arbitrary scalar field defined on $\mathscr{S}$ and where $\gamma_0 \in \mathscr{W}$ and $\alpha, \omega \in \mathbb{R}$; then we make use of (22) into $(21)_1$ and $(7)_3$ to obtain:

$$u(\mathbf{x}, \zeta ; t) = v(\zeta ; t) + \mathbf{x} \cdot \boldsymbol{\varphi}(\zeta ; t) + \omega(\zeta ; t)\Phi(\mathbf{x}), \tag{23}$$

where $\boldsymbol{\varphi} \in \mathscr{W}$ and $v, \omega \in \mathbb{R}$.

Together (20) and (23) lead to a representation for the motion field associated with the admissible strain which, for a prescribed function Φ, is parameterized on the three scalar fields v, θ and ω defined on $I \times [0,\tau)$ and on the two vector fields $\hat{\mathbf{v}}$, $\boldsymbol{\varphi} \in \mathscr{W}$ defined on $I \times [0,\tau)$. A concise representation for $\mathbf{u}$ is, therefore:

$$\mathbf{u}(\mathbf{x}, \zeta ; t) = \mathbf{v}(\zeta ; t) + \mathbf{A}(\zeta ; t)\mathbf{x} + \omega(\zeta ; t)\Phi(\mathbf{x})\mathbf{e}, \tag{24}$$

where

$$\mathbf{v} = \hat{\mathbf{v}} + v\mathbf{e}, \tag{25}$$

represents the *motion of the axis*, the *rigid rotation of the cross-section* is represented by the skew-symmetric tensor:

$$\mathbf{A} = \theta\mathbf{\Omega} + \mathbf{e} \otimes \boldsymbol{\varphi} - \boldsymbol{\varphi} \otimes \mathbf{e}, \tag{26}$$

here Ω is the skew-symmetric tensor such that $\mathbf{\Omega a} = \mathbf{e} \times \mathbf{a}$, $\forall \mathbf{a} \in \mathscr{V}$, and the term

$$u_{\Phi} = \omega\Phi. \tag{27}$$

represents the *warping* of the cross-section, with *amplitude* ω and where Φ is the *warping function*.

The admissible strain are thus obtained by making use of (20) and (23) into (7):

$$\begin{aligned} \gamma &= \hat{\mathbf{v}}_{,\zeta} + \boldsymbol{\varphi} + \theta_{,\zeta} \times \mathbf{e} \times \mathbf{x} + \omega\nabla_x\Phi, \\ \varepsilon &= v_{,\zeta} + \boldsymbol{\varphi}_{,\zeta} \cdot \mathbf{x} + \omega_{,\zeta}\phi; \end{aligned} \tag{28}$$

finally, by (13) and (19), we are led to the constitutive relation for the active stress $\mathbf{S}^{(A)}$:

$$\boldsymbol{\tau}^{(A)} = \mathbf{G}(\hat{\mathbf{v}}_{,\zeta} + \boldsymbol{\varphi} + \theta_{,\zeta}\mathbf{e} \times \mathbf{x} + \omega\nabla_{x}\Phi) + \mathbf{h}(v_{,\zeta} + \boldsymbol{\varphi}_{,\zeta} \cdot \mathbf{x} + \omega_{,\zeta}\Phi),$$
$$\sigma^{(A)} = \mathbf{h} \cdot (\hat{\mathbf{v}}_{,\zeta} + \boldsymbol{\varphi} + \theta_{,\zeta}\mathbf{e} \times \mathbf{x} + \omega\nabla_{x}\Phi) + E(v_{,\zeta} + \boldsymbol{\varphi}_{,\zeta} \cdot \mathbf{x} + \omega_{,\zeta}\Phi). \tag{29}$$

Remark 1. Hypothesis (14) was first introduced by Vlasov (1961), as one of the starting points for his theory of thin-walled beams; in §.**X** of Vlasov (1961) by using only this assumption a theory for beams of solid section is developed: however, the point of reactive stresses necessary to maintain such a constraint, as well as the restrictions (16) on the constitutive mapping are missing. In the resulting displacement field the representation (23) was postulated directly, rather than obtained as a necessary consequence of the kinematical compatibility equation once one assumes (14) [e.g. Reissner (1983a, b), and also Simo and Vu-Quoc (1991), eqn (10) which is obtained as the linearization of an *a priori* given finite deformation field].

3. DYNAMICS OF LINEAR ANISOTROPIC RODS

3.1. *Dynamical balance laws*

We define what we mean for an *elastic process* for a *thin rod-like body* $\mathscr{R}$: given a *body force* field $\mathbf{b}(\mathbf{x}, \zeta; t)$ on $\mathscr{R} \times [0, \tau)$, and a *surface force* field $\mathbf{s}(\mathbf{x}, \zeta; t)$ on $\mathscr{M} \times [0, \tau)$, the ordered array

$$p = [\mathbf{u}(\hat{\mathbf{v}}, v, \boldsymbol{\varphi}, \theta, \omega), \mathbf{E}(\mathbf{u}), \mathbf{S}^{(A)}(\mathbf{u}) + \mathbf{S}^{(R)}], \tag{30}$$

with $\mathbf{u}$ as in (24), $\mathbf{E}(\mathbf{u})$ as in (28), $\mathbf{S}^{(A)}$ as in (29) and $\mathbf{S}^{(R)}$ as in (18), is an *elastic process* corresponding to the *external force system* $(\mathbf{b}, \mathbf{s})$ if it verifies the equation of motion [cf. Gurtin (1972), §.60]:

$$\begin{aligned} \operatorname{div}\mathbf{S} + \mathbf{b} &= \rho\mathbf{u}_{,tt}, \quad \text{in } \mathscr{R} \times [0, \tau), \\ \mathbf{S}\mathbf{n} &= \mathbf{s}, \qquad \text{on } \mathscr{M} \times [0, \tau), \end{aligned} \tag{31}$$

where $\rho(\mathbf{x}, \zeta) > 0$ is the *density* field over $\mathscr{R}$ and $\mathbf{n}(\mathbf{x}, \zeta)$ is the normal unit vector to $\mathscr{M}$.

Let $\mathscr{A}$ the space of all elastic processes p for thin rods: for $\tau_0 < \tau$ we define the functional

$$\Psi\{p\} = \int_0^{\tau_0} \left(\frac{1}{2} \int_{\mathscr{R}} \mathbf{S} \cdot \mathbf{E} - \rho\mathbf{u}_{,t} \cdot \mathbf{u}_{,t} - \int_{\mathscr{R}} \mathbf{b} \cdot \mathbf{u} - \int_{\mathscr{M}} \mathbf{s} \cdot \mathbf{u} \right), \quad \forall p \in \mathscr{A}; \tag{32}$$

which, by the Hamilton–Kirchhoff principle [vid. Gurtin (1972), §.65], attains its minimum if p satisfies the equation of motion (31).

We take the first variation of (32), and taking into account (5), (10) and (13), we arrive at

$$0 = \int_0^{\tau_0} \int_0^{L} \left(\int_{\mathscr{S}} \boldsymbol{\tau}^{(A)} \cdot \boldsymbol{\gamma}_0 + \sigma^{(A)}\varepsilon_0 + \rho(\mathbf{u}_{,t} \cdot \mathbf{u}_{0,t}) + \mathbf{b} \cdot \mathbf{u}_0 + \int_{\partial\mathscr{S}} \mathbf{s} \cdot \mathbf{u}_0 \right), \tag{33}$$

where the variable $\mathbf{u}_0$ is represented by (20) and (23), and is related to γ_0 and ε_0 by (28).

As is customary with rod theories, we express the equation of motion in terms of resultants over the cross-section rather than in terms of tension. Accordingly, we define as usual the *stress resultant* $\mathbf{r} = \hat{\mathbf{r}} + r\mathbf{e}$ with the *shear* $\hat{\mathbf{r}}$ and the *normal force* r defined by:

$$\hat{\mathbf{r}}(\zeta;t) = \int_{\mathscr{S}} \boldsymbol{\tau}(\mathbf{x},\zeta;t), \quad r(\zeta;t) = \int_{\mathscr{S}} \sigma(\mathbf{x},\zeta;t), \tag{34}$$

and the *couple resultant* $\mathbf{m} = \hat{\mathbf{m}} + m\mathbf{e}$ where the *bending moment* $\hat{\mathbf{m}}$ and the *torque* m are given by:

$$m(\zeta;t)\mathbf{e} = \int_{\mathscr{S}} \mathbf{x} \times \boldsymbol{\tau}(\mathbf{x},\zeta;t), \quad \hat{\mathbf{m}}(\zeta;t) = \int_{\mathscr{S}} \mathbf{x} \times \mathbf{e}\sigma(\mathbf{x},\zeta;t); \tag{35}$$

moreover we need to introduce, for a given warping function Φ, the *bi-shear* r_Φ and the *bi-moment* m_Φ:

$$r_\Phi(\zeta;t) = \int_{\mathscr{S}} \nabla_x \Phi(\mathbf{x}) \cdot \boldsymbol{\tau}(\mathbf{x},\zeta;t), \quad m_\Phi(\zeta;t) = \int_{\mathscr{S}} \Phi(\mathbf{x})\sigma(\mathbf{x},\zeta;t), \tag{36}$$

and the loading terms:

$$\begin{aligned}
\hat{\mathbf{p}} &= \int_{\mathscr{S}} \hat{\mathbf{b}} + \int_{\partial\mathscr{S}} \hat{\mathbf{s}}, \quad p = \int_{\mathscr{S}} b + \int_{\partial\mathscr{S}} s, \\
\hat{\mathbf{c}} &= \int_{\mathscr{S}} b\mathbf{x} + \int_{\partial\mathscr{S}} s\mathbf{x}, \quad c = \mathbf{e} \cdot \int_{\mathscr{S}} \mathbf{x} \times \hat{\mathbf{b}} + \int_{\partial\mathscr{S}} \mathbf{x} \times \hat{\mathbf{s}}, \\
c_\Phi &= \int_{\mathscr{S}} b\Phi + \int_{\partial\mathscr{S}} s\Phi.
\end{aligned} \tag{37}$$

With these definitions, (33) becomes

$$\begin{aligned}
0 = &\int_0^{\tau 0} \int_0^L \hat{\mathbf{r}} \cdot \hat{\mathbf{v}}_{0,\zeta} + r v_{0,\zeta} + m\theta_{0,\zeta} + \mathbf{e} \times \hat{\mathbf{m}} \cdot \varphi_{0,\zeta} + r_\Phi \omega_0 + m_\Phi \omega_{0,\zeta} \\
&- \int_0^{\tau 0} \int_0^L \hat{\mathbf{p}} \cdot \hat{\mathbf{v}}_0 + p v_0 + \hat{\mathbf{c}} \cdot \varphi_0 + c\theta_0 + h\omega_0 \\
&- \int_0^{\tau 0} \int_0^L \rho(A\hat{\mathbf{v}}_{,t} \cdot \hat{\mathbf{v}}_{0,t} + (Av_{,t} + D_1 \omega_{,t}) v_{0,t} + (\mathbf{J}_{\varphi,t} + \mathbf{d}_1 \omega_{,t}) \cdot \varphi_{0,t} \\
&+ \operatorname{tr} \mathbf{J}^* \theta_{,t} \theta_{0,t} + (\mathbf{d}_1 \cdot \varphi_{,t} + D_1(\omega_{,t} + v_{,t}))\omega_{0,t}),
\end{aligned}$$

where the point $o \in \mathscr{S}$ is chosen in order to have:

$$\int_{\mathscr{S}} \mathbf{x} = \mathbf{0}, \tag{38}$$

$A > 0$ is the *area* of the cross-section $\mathscr{S}$, the positive definite *Euler* and *inertia tensors* $\mathbf{J}$ and $\mathbf{J}^*$ are defined as

$$\mathbf{J} = \int_{\mathscr{S}} \mathbf{x} \otimes \mathbf{x}, \quad \mathbf{J}^* = \int_{\mathscr{S}} (\mathbf{e} \times \mathbf{x}) \otimes (\mathbf{e} \times \mathbf{x}), \tag{39}$$

where:

$$D_1 = \int_{\mathscr{S}} \Phi, \quad \mathbf{d}_1 = \int_{\mathscr{S}} \mathbf{x}\Phi. \tag{40}$$

We assume without loss of generality that the density is constant, $\rho \equiv \rho_0$, and the material is homogeneous, $\mathbb{C}\equiv\mathbb{C}_0$; then by standard variational techniques we obtain the equations of motion on $I\times[0,\tau)$

$$\begin{aligned} r_{,\zeta}+p &= \rho(Av_{,tt}+D_1\omega_{,tt}),\\ \hat{\mathbf{r}}_{,\zeta}+\hat{\mathbf{p}} &= \rho A\hat{\mathbf{v}}_{,tt},\\ \mathbf{e}\times\hat{\mathbf{m}}_{,\zeta}-\hat{\mathbf{r}}-\hat{\mathbf{c}} &= \rho(\mathbf{J}\varphi_{,tt}+\mathbf{d}_1\omega_{,tt}),\\ m_{,\zeta}+c &= \rho\,\mathrm{tr}\,\mathbf{J}^*\theta_{,tt},\\ m_{\Phi,\zeta}-r_{\varphi}+c_{\varphi} &= \rho(D_1(\omega_{,tt}+v_{,tt})+\mathbf{d}_1\cdot\varphi_{,tt}), \end{aligned}\tag{41}$$

and the complementing boundary conditions on $\{0,L\}\times[0,\tau)$;

$$\mathbf{r}\cdot\mathbf{v}=0, \quad \mathbf{m}\cdot(\varphi\times\mathbf{e}+\theta\mathbf{e})=0, \quad m_{\Phi}\omega=0. \tag{42}$$

We use (29) in (34)–(36) to arrive at the constitutive relations for **r**, **m**, r_Φ and m_Φ:

$$\begin{aligned} r &= A\mathbf{h}\cdot(\boldsymbol{\varphi}+\hat{\mathbf{v}}_{,\zeta})+EAv_{,\zeta}+\mathbf{h}\cdot\mathbf{d}_2\omega+ED_1\omega_{,\zeta},\\ \hat{\mathbf{r}} &= A\mathbf{G}(\boldsymbol{\varphi}+\hat{\mathbf{v}}_{,\zeta})+A\mathbf{h}v_{,\zeta}+\mathbf{G}\mathbf{d}_2\omega+D_1\mathbf{h}\omega_{,\zeta},\\ m &= \mathbf{J}^*\cdot\mathbf{G}\theta_{,\zeta}+\mathbf{J}(\mathbf{h}\times e)\cdot\varphi_{,\zeta}+\mathbf{G}\cdot\mathbf{L}_1\omega+\mathbf{h}\times\mathbf{e}\cdot\mathbf{d}_1\omega_{,\zeta},\\ \mathbf{e}\times\hat{\mathbf{m}} &= E\mathbf{J}\boldsymbol{\varphi}_{,\zeta}+\mathbf{J}(\mathbf{h}\times\mathbf{e})\theta_{,\zeta}+E\mathbf{d}_1\omega_{,\zeta}+\mathbf{L}_1\mathbf{h}\omega,\\ r_\Phi &= \mathbf{G}\mathbf{d}_2\cdot(\varphi+\hat{\mathbf{v}}_{,\zeta})+\mathbf{G}\cdot\mathbf{L}_1\theta_{,\zeta}+\mathbf{G}\cdot\mathbf{L}_2\omega+\mathbf{d}_2\cdot\mathbf{h}v_{,\zeta}+\mathbf{L}_1\mathbf{h}\cdot\boldsymbol{\varphi}_{,\zeta}+\mathbf{h}\cdot\mathbf{d}_3\omega_{,\zeta},\\ m_\Phi &= D_1\mathbf{h}\cdot(\boldsymbol{\varphi}+\hat{\mathbf{v}}_{,\zeta})+\mathbf{h}\times\mathbf{e}\cdot\mathbf{d}_1\theta_{,\zeta}+\mathbf{h}\cdot\mathbf{d}_3\omega+ED_1v_{,\zeta}+E\mathbf{d}_1\cdot\boldsymbol{\varphi}_{,\zeta}+ED_2\omega_{,\zeta}, \end{aligned}\tag{43}$$

where

$$\begin{aligned} D_2 &= \int_{\mathscr{S}} \Phi^2,\\ \mathbf{d}_2 = \int_{\mathscr{S}} \nabla_{\mathbf{x}}\Phi, &\quad \mathbf{d}_3 = \frac{1}{3}\int_{\mathscr{S}} \nabla_{\mathbf{x}}(\Phi)^2,\\ \mathbf{L}_1 = \int_{\mathscr{S}} \mathbf{x}\otimes\nabla_{\mathbf{x}}\Phi, &\quad \mathbf{L}_2 = \int_{\mathscr{S}} \nabla_{\mathbf{x}}\Phi\otimes\nabla_{\mathbf{x}}\Phi. \end{aligned}\tag{44}$$

The equations of motion for thin anisotropic rods in terms of the unknown fields $\hat{\mathbf{v}}$, v, $\boldsymbol{\varphi}$, θ and ω on $I\times[0,\tau)$ follow, therefore, from (41) and (43):

$$\begin{aligned} &A\mathbf{h}\cdot(\boldsymbol{\varphi}_{,\zeta}+\hat{\mathbf{v}}_{,\zeta\zeta})+EAv_{,\zeta\zeta}+ED_1\omega_{,\zeta\zeta}+\mathbf{h}\cdot\mathbf{d}_2\omega_{,\zeta}+p = \rho(Av_{,tt}+D_1\omega_{,tt}),\\ &A\mathbf{G}(\boldsymbol{\varphi}_{,\zeta}+\mathbf{v}_{,\zeta\zeta})+A\mathbf{h}v_{,\zeta\zeta}+D_1\mathbf{h}\omega_{,\zeta\zeta}+\mathbf{G}\mathbf{d}_2\omega_{,\zeta}+\hat{\mathbf{p}} = \rho A\hat{\mathbf{v}}_{,tt},\\ &E\mathbf{J}\boldsymbol{\varphi}_{,\zeta\zeta}+\mathbf{J}(\mathbf{h}\times\mathbf{e})\theta_{,\zeta\zeta}+E\mathbf{d}_1\omega_{,\zeta\zeta}+(\mathbf{L}_1\mathbf{h}-D_1\mathbf{h})\omega_{,\zeta}\\ &\qquad -A\mathbf{G}\hat{\mathbf{v}}_{,\zeta}-A\mathbf{h}v_{,\zeta}-A\mathbf{G}\varphi-\mathbf{G}\mathbf{d}_2\omega-\hat{\mathbf{c}} = \rho(\mathbf{J}\boldsymbol{\varphi}_{,tt}+\mathbf{d}_1\omega_{,tt}),\\ &\mathbf{J}^*\cdot\mathbf{G}\theta_{,\zeta\zeta}+\mathbf{J}(\mathbf{h}\times\mathbf{e})\cdot\boldsymbol{\varphi}_{,\zeta\zeta}+\mathbf{h}\times\mathbf{e}\cdot\mathbf{d}_1\omega_{,\zeta\zeta}+\mathbf{G}\cdot\mathbf{L}_1\omega_{,\zeta}+c = \rho\,\mathrm{tr}\,\mathbf{J}^*\theta_{,tt},\\ &D_1\mathbf{h}\cdot\hat{\mathbf{v}}_{,\zeta\zeta}+ED_2\omega_{,\zeta\zeta}+\mathbf{h}\times\mathbf{e}\cdot\mathbf{d}_1\theta_{,\zeta\zeta}+ED_1v_{,\zeta\zeta}+E\mathbf{d}_1\cdot\boldsymbol{\varphi}_{,\zeta\zeta}\\ &\qquad -\mathbf{G}\mathbf{d}_2\cdot\hat{\mathbf{v}}_{,\zeta}-\mathbf{G}\cdot\mathbf{L}_1\theta_{,\zeta}+(D_1\mathbf{h}-\mathbf{L}_1\mathbf{h})\cdot\boldsymbol{\varphi}_{,\zeta}-\mathbf{d}_2\cdot\mathbf{h}v_{,\zeta}\\ &\qquad\qquad -\mathbf{G}\cdot\mathbf{L}_2\omega-\mathbf{G}\mathbf{d}_2\cdot\boldsymbol{\varphi}+c_\Phi = \rho(D_1(\omega_{,tt}+v_{,tt})+\mathbf{d}_1\cdot\varphi_{,tt}). \end{aligned}\tag{45}$$

The reactive stress $\mathbf{S}^{(R)}$ can be calculated from the equation of motion (31), once we solve (45); indeed, by putting (10) in (31) we arrive at:

$$\begin{aligned} \operatorname{div}_x \hat{\mathbf{S}} + \boldsymbol{\tau}_{,z} + \hat{\mathbf{b}} &= \rho \hat{\mathbf{u}}_{,tt}, \quad \text{in } \mathscr{R} \times [0, \tau), \\ \hat{\mathbf{S}}\mathbf{n} &= \hat{\mathbf{s}}, \qquad \text{on } \mathscr{M} \times [0, \tau), \end{aligned} \tag{46}$$

and

$$\begin{aligned} \operatorname{div}_x \boldsymbol{\tau} + \sigma_{,z} + b &= \rho u_{,tt}, \quad \text{in } \mathscr{R} \times [0, \tau), \\ \boldsymbol{\tau} \cdot \mathbf{n} &= s, \qquad \text{on } \mathscr{M} \times [0, \tau), \end{aligned} \tag{47}$$

where div_x denotes the divergence operator with respect to $\mathbf{x}$. Then, since (18) holds, we have:

$$\begin{aligned} \operatorname{div}_x \mathbf{S}^{(R)} &= -(\mathbf{G}\gamma + \mathbf{h}\varepsilon)_{,z} + \rho \hat{\mathbf{u}}_{,tt} - \hat{\mathbf{b}}, \quad \text{in } \mathscr{R} \times [0, \tau), \\ \mathbf{S}^{(R)}\mathbf{n} &= \hat{\mathbf{s}}, \qquad \text{on } \mathscr{M} \times [0, \tau), \end{aligned}$$

with γ, ε and $\hat{\mathbf{u}}$ given, respectively, by (28) and (20).

3.2. *The warping function*

In the literature [e.g. Vlasov (1961), Reissner (1983a), Simo and Vu-Quoc (1991)] the warping function is determined by considering the Saint-Venant's uniform torsion problem for a prismatic body $\mathscr{R} \equiv \mathscr{S} \times I$:

$$\begin{aligned} \Delta_x \Phi &= 0, \quad \text{in } \mathscr{S}, \\ (\nabla_x \Phi + \mathbf{e} \times \mathbf{x}) \cdot \mathbf{n} &= 0, \quad \text{on } \partial\mathscr{S}, \end{aligned} \tag{48}$$

here Δ_x denotes the two-dimensional Laplace operator; some authors require also that Φ obeys other conditions, as in Vlasov (1961) where it is required to obey the so-called *orthogonality conditions*:

$$D_1 = 0, \quad \mathbf{d}_1 = \mathbf{0}. \tag{49}$$

As a matter of fact, Φ is independent of t and it makes sense to assume the warping function as the solution of a static equilibrium problem; moreover, since many rod theories are based on some kind of approximations of the Saint-Venant's beam theory [e.g. Antman (1972), §§.11–12], the fact that in such a theory $\mathbf{b} = \mathbf{0}$ and $\hat{\mathbf{s}} = \mathbf{0}$ is still regarded as a reasonable approximation.

The equilibrium equation we arrive at from (45), by setting all the dynamical terms to zero, are derived on the contrary from a three-dimensional constrained elasticity problem which is different from the Saint-Venant's one: i.e. find an elastic state

$$p = [\mathbf{u}, \mathbf{E}(\mathbf{u}), \mathbf{S}^{(A)}(\mathbf{u}) + \mathbf{S}^{(R)}] \tag{50}$$

with $\mathbf{u}$ as in (24), $\mathbf{E}(\mathbf{u})$ as in (28), $\mathbf{S}^{(A)}$ as in (29) and $\mathbf{S}^{(R)}$ as in (18), which verifies

$$\begin{aligned} \operatorname{div} \mathbf{S} + \mathbf{b} &= \mathbf{0}, \quad \text{in } \mathscr{R}, \\ \mathbf{S}\mathbf{n} &= \mathbf{s}, \quad \text{on } \mathscr{M}, \end{aligned} \tag{51}$$

for a given external force system $(\mathbf{b}, \mathbf{s})$; therefore, we search the warping function as solution for the three-dimensional constrained elasticity problem (50), (51), by assuming the external force system as in the Saint-Venant's problem and the semi-inverse assumption on the stress tensor:

$$\mathbf{S}_{,\zeta\zeta} = \mathbf{0}, \tag{52}$$

which was originally proposed by Voigt (1910) and was used in Daví and Tiero (1994) to solve the Saint-Venant's problem for anisotropic solids.

If the constant elasticity tensor is invertible, then (52) is equivalent to:

$$\mathbf{E}_{,\zeta\zeta} = \operatorname{sym}(\nabla_x \mathbf{u})_{,\zeta\zeta} = \operatorname{sym}\nabla_x(\mathbf{u}_{,\zeta\zeta}) = \mathbf{0}, \tag{53}$$

which implies:

$$\mathbf{u}_{,\zeta\zeta} = \mathbf{a}_0 + \mathbf{w}_0 \times \mathbf{p}, \tag{54}$$

where $\mathbf{a}_0$ and $\mathbf{w}_0$ are two constant vectors. By integrating (54) twice along ζ we get:

$$\mathbf{u}(\mathbf{x},\zeta) = \mathbf{a}_2(\mathbf{x}) + \zeta\mathbf{a}_1(\mathbf{x}) + \tfrac{1}{2}\zeta^2(\mathbf{a}_0 + \mathbf{w}_0 \times \mathbf{x}) + \tfrac{1}{6}\zeta^3\mathbf{w}_0 \times \mathbf{e}, \tag{55}$$

where, in order to comply with the prescribed internal constraint (14), we must have:

$$\mathbf{a}_\alpha(\mathbf{x}) = \hat{\mathbf{b}}_\alpha + \omega_\alpha \mathbf{e} \times \mathbf{x} + \psi_\alpha(\mathbf{x}), \quad \hat{\mathbf{b}}_\alpha \in \mathscr{W}, \quad w_\alpha \in \mathbb{R}, \quad \alpha = 1,2, \tag{56}$$

where $\hat{\mathbf{b}}_\alpha$ and w_α are constant; we use (3) and (7) and obtain the strain associated with (14) and (52):

$$\begin{aligned} \gamma(\mathbf{x},\zeta) &= \nabla_x\psi_2(\mathbf{x}) + \hat{\mathbf{b}}_2 + w_2\mathbf{e} \times \mathbf{x} + \zeta(\nabla_x\psi_1 + \hat{\mathbf{a}}_0 + w_0\mathbf{e} \times \mathbf{x}), \\ \varepsilon(\mathbf{x},\zeta) &= \psi_1(\mathbf{x}) + \zeta(\alpha_0 + \hat{\mathbf{w}}_0 \times \mathbf{x} \cdot \mathbf{e}), \end{aligned} \tag{57}$$

where $\mathbf{a}_0 = \hat{\mathbf{a}}_0 + a_0\mathbf{e}$ and $\mathbf{w}_0 = \hat{\mathbf{w}}_0 + w_0\mathbf{e}$.

On putting (57) in conjunction with the constitutive relations (13), (19) and the definitions (34), (35), into the equilibrium equations:

$$\mathbf{r}_{,\zeta} = \mathbf{0}, \quad \mathbf{m}_{,\zeta} + \mathbf{e} \times \mathbf{r} = \mathbf{0}, \tag{58}$$

we arrive at the conditions:

$$\begin{aligned} w_0 &= 0, \quad \mathbf{a}_0 = \mathbf{0}, \\ \psi_1 &= \tfrac{1}{2}\mathbf{G}^{-1}\mathbf{h} \otimes (\mathbf{w}_0 \times \mathbf{e}) \cdot \mathbf{x} \otimes \mathbf{x} + c_1, \quad c_1 = \text{const.}; \end{aligned} \tag{59}$$

finally, we use (13), (19), (57) and (59), in the equilibrium equation obtained by setting zero the dynamical terms into (47) and upon the definition of:

$$\nabla_x\Phi(\mathbf{x}) = w_2^{-1}(\nabla_x\psi_2(\mathbf{x}) + (\mathbf{G}^{-1}\mathbf{h} \otimes \mathbf{G}^{-1}\mathbf{h})(\mathbf{x} \otimes \mathbf{x})(\mathbf{w}_0 \times \mathbf{e}) + \hat{\mathbf{b}}_2), \tag{60}$$

we arrive at

$$\begin{aligned} \operatorname{div}_x \mathbf{G}\nabla_x\Phi &= 0, \quad \text{in } \mathscr{S}, \\ \mathbf{G}(\nabla_x\Phi + \mathbf{e} \times \mathbf{x}) \cdot \mathbf{n} &= 0, \quad \text{on } \partial\mathscr{S}. \end{aligned} \tag{61}$$

The boundary value problem (61) plays the role of (48) in the three-dimensional constrained elasticity problem (50) and (51), and reduces to it for *isotropic* rods where $\mathbf{G} = G\hat{\mathbf{I}}$, where $\hat{\mathbf{I}}$ denotes the identity in $\mathscr{X}$.

Remark 2. If the warping function Φ obeys (61), then we have the following identity:

$$\mathbf{G}\cdot\int_{\mathscr{S}}\nabla_{\mathbf{x}}\Phi\otimes\Phi=\int_{\mathscr{S}}\Phi\mathbf{G}\nabla_{\mathbf{x}}\Phi\cdot\mathbf{n}=-\int_{\partial\mathscr{S}}\Phi\mathbf{G}(\mathbf{e}\times\mathbf{x})\cdot\mathbf{n}$$

$$=-\int_{\mathscr{S}}\nabla_{\mathbf{x}}\Phi\cdot\mathbf{G}(\mathbf{e}\times\mathbf{x})=-\mathbf{G}\cdot\mathbf{\Omega}\int_{\mathscr{S}}\mathbf{x}\otimes\nabla_{\mathbf{x}}\Phi,$$

which leads to:

$$\mathbf{G}\cdot\mathbf{L}_2=-\mathbf{G}\cdot\mathbf{\Omega}\mathbf{L}_1; \tag{62}$$

moreover, since (61) implies:

$$\int_{\mathscr{S}}\mathbf{G}(\nabla_{\mathbf{x}}\Phi+\mathbf{e}\times\mathbf{x})=0,$$

then, when (38) holds, we have:

$$\mathbf{d}_2=\mathbf{0}. \tag{63}$$

3.3. *Monocline rods*

In a rod-like body we can recognize a "global" symmetry which is based, roughly speaking, upon the structure of cartesian product between a cross-section and an axis of the underlying rod-like region; we can chose the material in such a way it recollects this "global" symmetry with the local one described by its symmetry group: this happens, for instance, when $\mathscr{G}_{\mathrm{p}}$ contains all the rotations of amplitude π around $\mathbf{e}$:

$$\mathbf{Q}_{\mathbf{e}}^{\pi}=2\mathbf{e}\otimes\mathbf{e}-\mathbf{I}, \tag{64}$$

composed with the central reflection $\mathbf{Q}=-\mathbf{I}$; such a material is called *monocline* with respect to $\mathbf{e}$.

In this section we shall assume the rod to be comprised of a constrained monocline material whose axis of anisotropy coincides with the axis of the rod itself. According to Gurtin (1972; §.20), the constitutive relation for such a material can be obtained by setting:

$$\mathbb{L}=\mathbf{0},\quad \mathbf{h}=\mathbf{0}, \tag{65}$$

into (13); moreover, conditions (19) hold and, therefore, the constitutive relations (43) reduce to:

$$\begin{aligned}
r&=EAv_{,\zeta}+ED_1\omega_{,\zeta},\\
\hat{\mathbf{r}}&=A\mathbf{G}(\boldsymbol{\varphi}+\hat{\mathbf{v}}_{,\zeta})+\mathbf{G}\mathbf{d}_2\omega,\\
m&=\mathbf{J}^*\cdot\mathbf{G}\theta_{,\zeta}+\mathbf{G}\cdot\mathbf{L}_1\omega,\\
\mathbf{e}\times\hat{\mathbf{m}}&=E\mathbf{J}_{\varphi,\zeta}+E\mathbf{d}_1\omega_{,\zeta},\\
r_\Phi&=\mathbf{G}\mathbf{d}_2(\varphi+\hat{\mathbf{v}}_{,\zeta})+\mathbf{G}\cdot\mathbf{L}_1\theta_{,\zeta}+\mathbf{G}\cdot\mathbf{L}_2\omega,\\
m_\phi&=ED_1v_{,\zeta}+E\mathbf{d}_1\cdot\varphi_{,\zeta}+ED_2\omega_{,\zeta}.
\end{aligned} \tag{66}$$

whereas (45) become:

$$
\begin{aligned}
EAv_{,\zeta\zeta} + ED_1\omega_{,\zeta\zeta} + p &= \rho(Av_{,tt} + D_1\omega_{,tt}),\\
A\mathbf{G}(\boldsymbol{\varphi}_{,\zeta} + \hat{\mathbf{v}}_{,\zeta\zeta}) + \mathbf{G}\mathbf{d}_2\omega_{,\zeta} + \hat{\mathbf{p}} &= \rho A\hat{\mathbf{v}}_{,tt},\\
E\mathbf{J}\boldsymbol{\varphi}_{,\zeta\zeta} + E\mathbf{d}_1\omega_{,\zeta\zeta} - A\mathbf{G}(\boldsymbol{\varphi} + \hat{\mathbf{v}}_{,\zeta}) - \mathbf{G}\mathbf{d}_2\omega_{,\zeta} - \hat{\mathbf{c}} &= \rho(\mathbf{J}\boldsymbol{\varphi}_{,tt} + \mathbf{d}_1\omega_{,tt}),\\
\mathbf{J}^*\cdot\mathbf{G}\theta_{,\zeta\zeta} + \mathbf{G}\cdot\mathbf{L}_1\omega_{,\zeta} + c &= \rho\,\mathrm{tr}\,\mathbf{J}^*\theta_{,tt},\\
ED_1v_{,\zeta\zeta} + E\mathbf{d}_1\cdot\boldsymbol{\varphi}_{,\zeta\zeta} + ED_2\omega_{,\zeta\zeta} - \mathbf{G}\mathbf{d}_2\cdot(\boldsymbol{\varphi} + \hat{\mathbf{v}}_{,\zeta}) - \mathbf{G}\cdot\mathbf{L}_1\theta_{,\zeta}&\\
- \mathbf{G}\cdot\mathbf{L}_2\omega + c_{\Phi} &= \rho(D_1(\omega_{,tt} + v_{,tt}) + \mathbf{d}_1\cdot\varphi_{,tt}). \quad (67)
\end{aligned}
$$

When the orthogonality conditions (49) hold, the constitutive relation $(66)_6$ is the same proposed by Vlasov (1961):

$$m_{\Phi} = ED_2\omega_{,\zeta}, \tag{68}$$

the constant D_2 being referred by some authors as the *Vlasov constant*; furthermore, if $(49)_1$ holds, the equations of motion (67) generalize the equations given in Reissner (1983a) for isotropic rods.

Let the warping function be a solution of (61) which obeys the orthogonality conditions (49): then the equations of motion (67) splits in three groups, the first one concerned with the *flexure* of the rod and which involves bending moment, shears, and the associated kinematical variables $\boldsymbol{\varphi}$ and $\hat{\mathbf{v}}$:

$$
\begin{aligned}
A\mathbf{G}(\boldsymbol{\varphi}_{,\zeta} + \hat{\mathbf{v}}_{,\zeta\zeta}) + \hat{\mathbf{p}} &= \rho A\hat{\mathbf{v}}_{,tt},\\
E\mathbf{J}\boldsymbol{\varphi}_{,\zeta\zeta} - A\mathbf{G}(\boldsymbol{\varphi} + \hat{\mathbf{v}}_{,\zeta}) - \hat{\mathbf{c}} &= \rho\mathbf{J}\boldsymbol{\varphi}_{,tt},
\end{aligned} \tag{69}
$$

the second one concerned with the *torsion-warping* deformation which involves torque, bi-shear, bi-moment and ω, θ:

$$
\begin{aligned}
\mathbf{J}^*\cdot\mathbf{G}\theta_{,\zeta\zeta} + \mathbf{G}\cdot\mathbf{L}_1\omega_{,\zeta} + c &= \rho\,\mathrm{tr}\,\mathbf{J}^*\theta_{,tt},\\
ED_2\omega_{,\zeta\zeta} - \mathbf{G}\cdot\mathbf{L}_1\theta_{,\zeta} + \mathbf{G}\cdot\boldsymbol{\Omega}\mathbf{L}_1\omega + c_{\Phi} &= 0,
\end{aligned} \tag{70}
$$

and the last one concerned with the extension and involving only the normal force and v:

$$EAv_{,\zeta\zeta} + p = \rho Av_{,tt}. \tag{71}$$

From (69) we obtain the following equation for $\hat{\mathbf{v}}$:

$$E\mathbf{J}\hat{\mathbf{v}}_{,\zeta\zeta\zeta\zeta} - \rho\mathbf{J}(\hat{\mathbf{I}} + E\mathbf{G}^{-1})\hat{\mathbf{v}}_{,\zeta\zeta tt} + \rho A\hat{\mathbf{v}}_{,tt} + \rho^2\mathbf{J}\mathbf{G}^{-1}\hat{\mathbf{v}}_{,tttt} + \mathbf{j} = \mathbf{0}, \tag{72}$$

with $\mathbf{j} = A\mathbf{J}\mathbf{G}^{-1}(E\hat{\mathbf{p}}_{,\zeta\zeta} - \rho\hat{\mathbf{p}}_{,tt}) + \hat{\mathbf{p}} - \mathbf{c}_{,\zeta}$.

When the material comprising the rod is *orthotropic-rhombic*, which means that its symmetry group contains the reflection on three mutually orthogonal directions $\mathbf{e}_\alpha(\alpha = 1{,}2)$, $\mathbf{e}_3 = \mathbf{e}$:

$$\mathbf{Q}^{\pi}_{\mathbf{e}_i} = 2\mathbf{e}_i \otimes \mathbf{e}_i - \mathbf{I}. \quad (i = 1, 2, 3, \text{no sum})$$

then

$$\mathbf{G} = G_1\mathbf{e}_1 \otimes \mathbf{e}_1 + G_2\mathbf{e}_2 \otimes \mathbf{e}_2,$$

and (72) reduces to eqn (53) of Green *et al.* (1967), which is however obtained within the context of a director theory; furthermore, when the rod is *isotropic* and in absence of applied forces, we obtain from (72) the equation of Timoshenko's (1921) beam theory [see also Volterra (1955), Medick (1966) and Antman (1972) where a survey of earlier rod theories which lead to this equation are given].

Finally, eqns (70) can be solved to obtain the following one for θ:

$$ED_2(\mathbf{J}^* \cdot \mathbf{G})\theta_{,\zeta\zeta\zeta\zeta} - \rho ED_2 \operatorname{tr} \mathbf{J}^*\theta_{,\zeta\zeta tt} + ((\mathbf{G} \cdot \mathbf{L}_1)^2 - \mathbf{G} \cdot \mathbf{\Omega L}_1(\mathbf{J}^* \cdot \mathbf{G}))\theta_{,\zeta\zeta} - \rho \mathbf{G} \cdot \mathbf{\Omega L}_1 \operatorname{tr} \mathbf{J}^*\theta_{,tt} + j = 0, \quad (73)$$

where $j = ED_2 c_{,\zeta\zeta} + \mathbf{G} \cdot (\mathbf{\Omega L}_1 c - \mathbf{L}_1 c_{\mathbf{\Phi},\zeta})$; this equation has no counterpart in Green *et al.* (1967).

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